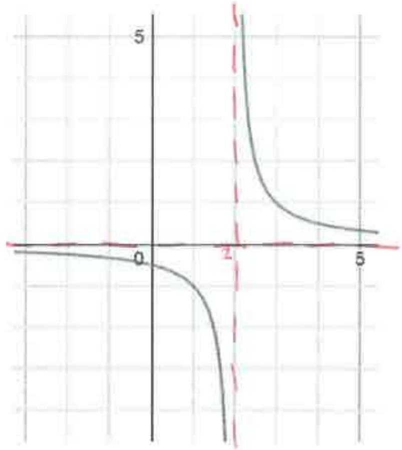
Reciprocal Functions

Chapter 8 Notes.

From the previous chapter on rational functions we saw that non-permissible values provide the location of asymptotes on the corresponding graph.

e.g. The function $\frac{1}{x-2}$ has the non-permissible value $x = 2$, the graph of $y = \frac{1}{x-2}$ therefore has an asymptote at $x = 2$.



As $x \to \infty$, $\frac{1}{x-2} \to 0^+$

As $x \to -\infty$, $\frac{1}{x-2} \to 0^-$

As $x \to 2^+$, $\frac{1}{x-2} \to +\infty$

As $x \to 2^-$, $\frac{1}{x-2} \to -\infty$

Notice there is also a horizontal asymptote at $y = 0$. This is because $\frac{1}{x-2} \neq 0$ for any value of x.

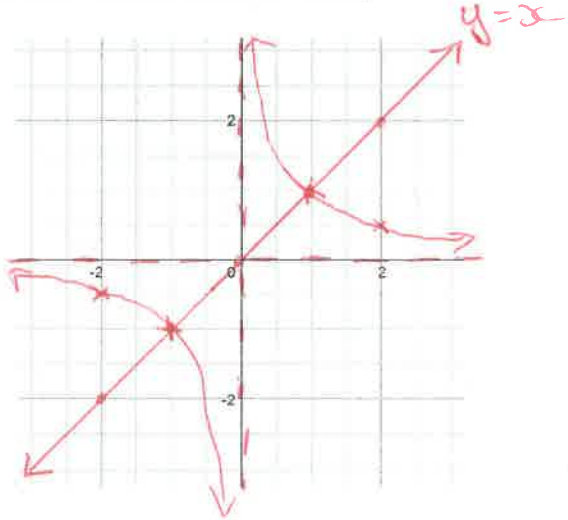
If $y = f(x)$ then the reciprocal function is $y = \frac{1}{f(x)}$.

Graphing Reciprocal Functions

Graph $y = x$ and its reciprocal function using a table of values:

$y = x$

x	y
-2	-2
-1	-1
0	0
1	1
2	2



$y = \frac{1}{x}$

x	y
-2	$-\frac{1}{2}$
-1	-1
0	N.P. Vertical Asym.
1	1
2	$\frac{1}{2}$

$\frac{1}{x} \neq 0$ so HA at $y=0$.

What points do they have in common? (-1,-1), (1,1).

If $f(x) = 1$ then $\frac{1}{f(x)} = \frac{1}{1} = 1$ and if $f(x) = -1$ then $\frac{1}{f(x)} = \frac{1}{-1} = -1$.

Thus the points on the graph which intersect with the lines $y = 1$ and $y = -1$ will also be points on the graph of the reciprocal function.

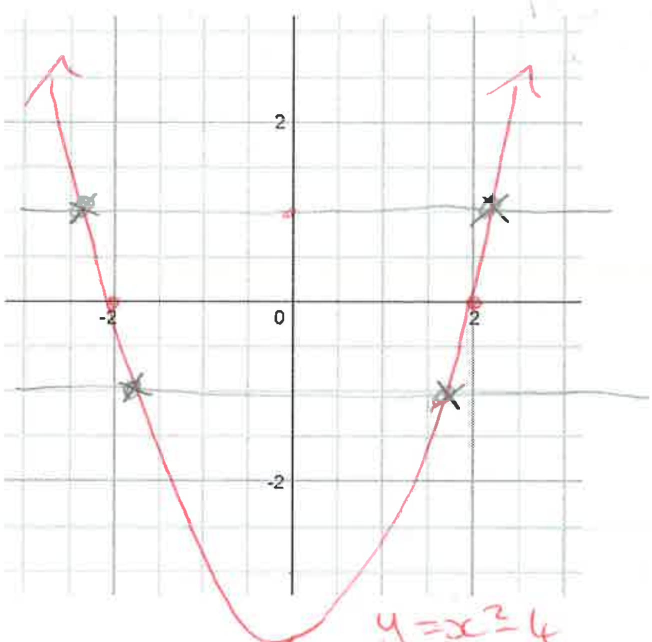
What else do you notice?

The x-intercept(s) of the graph of $y = f(x)$ give the location of the asymptotes of the graph of the reciprocal function $y = \frac{1}{f(x)}$.

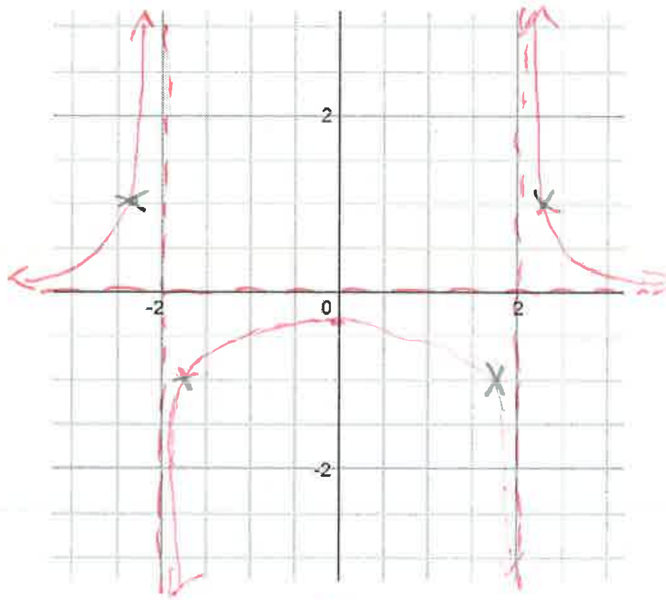
There is a horizontal asymptote at $y = 0$ on the graph of the reciprocal function as $\frac{1}{f(x)} \neq 0$.

Example: Graph $y = x^2 - 4$ and its reciprocal function.

vertex $(0, -4)$



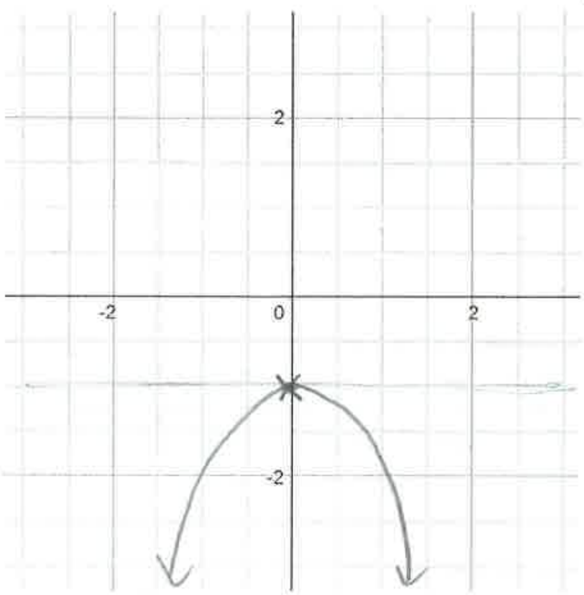
$y = x^2 - 4$



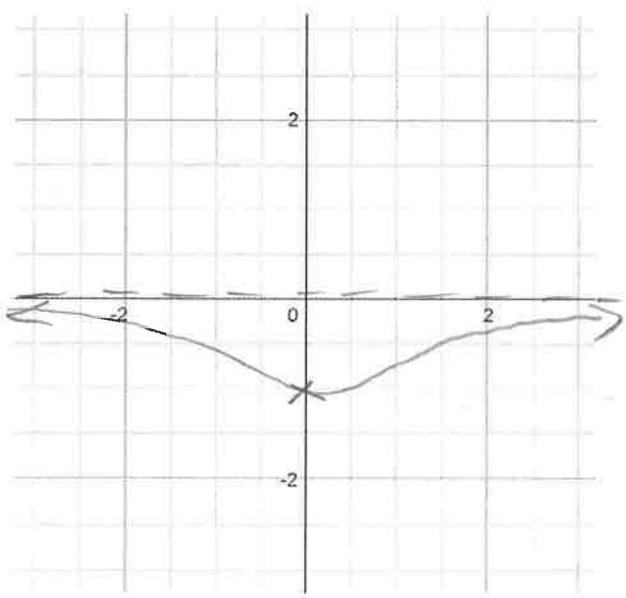
$y = \frac{1}{x^2 - 4}$

Example: Graph $y = -2x^2 - 1$ and its reciprocal.

$v(0, -1)$ $\cap$ $a, q < 0$ so no x-int $\Rightarrow$ no vertical



$y = -2x^2 - 1$



$y = \frac{1}{-2x^2 - 1}$

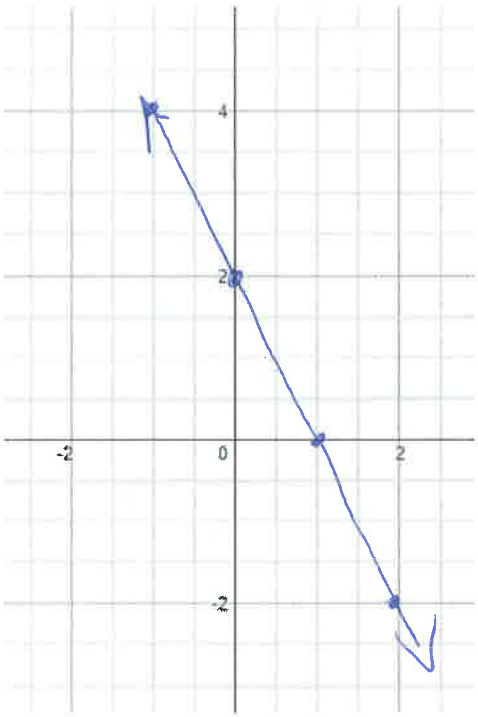
Workbook 8.3 - 8.5 exercises only.

Absolute Value Functions and Equations

Example: Graph $y = -2x + 2$ and $y = |-2x + 2|$ using a table of values.

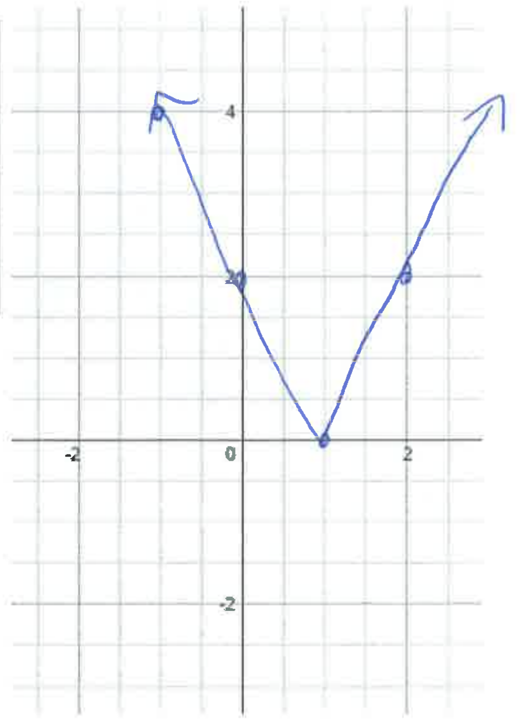
$y = -2x + 2$

x	y
-2	6
-1	4
0	2
1	0
2	-2
3	-4



$y = |-2x + 2|$

x	y
-2	6
-1	4
0	2
1	0
2	2
3	4



What do you notice?

The graph of $y = |f(x)|$ is the graph of $y = f(x)$ with any part below the x-axis ($y < 0$) reflected in the x-axis.

We can express absolute value functions using piecewise notation. We break the function $y = |f(x)|$ into two parts, the part for which $f(x) \geq 0$ and the part for which $f(x) < 0$:

For $y = |-2x + 2|$, $y = -2x + 2$ when $-2x + 2 \geq 0$ and $y = -(-2x + 2)$ when $-2x + 2 < 0$.

$$\begin{aligned} -2x + 2 &\geq 0 \\ -2x &\geq -2 \\ x &\leq 1 \end{aligned} \qquad\qquad \begin{aligned} -2x + 2 &< 0 \\ -2x &< -2 \\ x &> 1 \end{aligned}$$

Thus $y = |-2x + 2|$ can be written as the piecewise function:

$$y = \begin{cases} -2x + 2 & if \quad x \leq 1 \\ 2x - 2 & if \quad x > 1 \end{cases}$$

There can be at most two pieces if $f(x)$ is a linear function. How many pieces can there be if $f(x)$ is a quadratic function? at most 3

Example: Graph $y = |x^2 - 5x + 4|$ and write in piecewise notation.

Start by graphing $y = x^2 - 5x + 4$, then reflect the piece for which $y < 0$ in the x-axis.

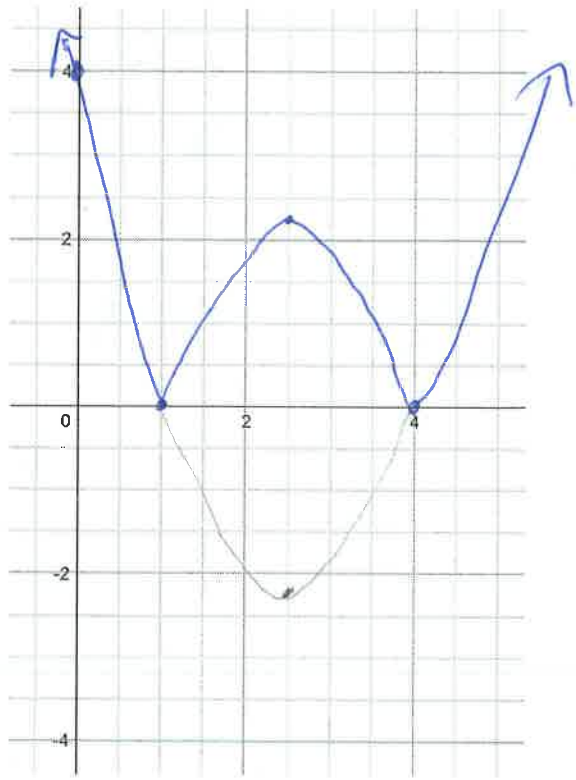
$y = (x-4)(x-1)$

x-intercepts at 1, 4

$a > 0$ ↻

$x = 0 \Rightarrow y = 4$

vertex, $x = 2.5 \Rightarrow y = -2.25$



Piecewise notation:

$$y = \begin{cases} x^2 - 5x + 4 & if \quad x \le 1 \text{ or } x \geq 4 \\ -x^2 + 5x - 4 & if \quad 1 < x < 4 \end{cases}$$

Solving Absolute Value Equations

To solve $|f(x)| = g(x)$ graphically, graph $y = |f(x)|$ and $y = g(x)$ on the same grid and find the points of intersection.

To solve algebraically consider the absolute value function piecewise.

Linear Absolute Value Equations

Example: Solve $|4x - 7| = 2 + x$.

$4x - 7 = 2 + x$ if $4x - 7 \geq 0$ and $-(4x - 7) = 2 + x$ if $4x - 7 < 0$

$3x = 9$ if $4x \geq 7$

$x = 3$ if $x \geq \frac{7}{4}$

$3 \geq \frac{7}{4}$ so $x = 3$ is a solution

$-4x + 7 = 2 + x$ if $4x < 7$

$-5x = -5$ $\quad x < \frac{7}{4}$

$x = 1$

$1 < \frac{7}{4}$

so $x = 1$ is a solution.

* show on Desmos.com

Quadratic Absolute Value Equations

Example: Solve $|x^2 - 3x - 11| - 3 = 4$.

Simplify first $|x^2 - 3x - 11| = 7$.

Now $x^2 - 3x - 11 = 7$ if $x^2 - 3x - 11 \geq 0$ and $-(x^2 - 3x - 11) = 7$ if $x^2 - 3x - 11 < 0$

$x^2-3x-11=7$

$x^2-3x-18=0$

$(x-6)(x+3)=0$

$x=-3,6$

IF $x^2-3x-11\geq 0$

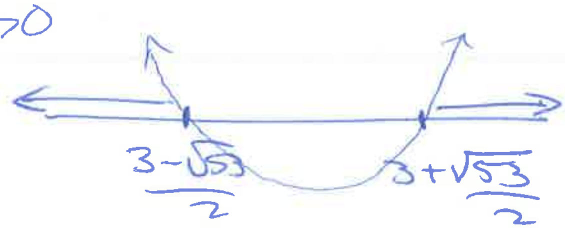
Critical values:

$x^2-3x-11=0$

$x=\frac{3\pm\sqrt{9+44}}{2}$

$=\frac{3\pm\sqrt{53}}{2}$

$a>0$



so IF $x\leq\frac{3-\sqrt{53}}{2}$ or $x\geq\frac{3+\sqrt{53}}{2}$

≈ -2.1 ≈ 5.1

Workbook 8.1 and 8.2

$-(x^2-3x-11)=7$

$x^2-3x-4=0$

$(x-4)(x+1)=0$

$x=-1,4$

IF $x^2-3x-11<0$

So IF $\frac{3-\sqrt{53}}{2}<x<\frac{3+\sqrt{53}}{2}$

$\frac{3-\sqrt{53}}{2}<-1,4<\frac{3+\sqrt{53}}{2}$

So $x=-1,4$ are solutions

$x=-3,-1,4,6$ are all solutions.

$-3\leq-2.1$ and $6\geq5.1$

So $x=-3,6$ are sols.

* show on Desmos

Recap:

1. If (a, b) is a point on $y = f(x)$ then $\left(a, \frac{1}{b}\right)$ is a point on the reciprocal funciton $y = \frac{1}{f(x)}$.

2. The points at which $y = f(x)$ intersects with the lines $y = 1$ and $y = -1$ are also points on the reciprocal function $y = \frac{1}{f(x)}$.

3. The x-intercepts of $y = f(x)$ give the location of the asymptotes of the graph of $y = \frac{1}{f(x)}$.

4. $y = \frac{1}{f(x)}$ will always have a horizontal asymptote at $y = 0$.

5. Check your graph. As $x \to \pm\infty$ what happens to $\frac{1}{f(x)}$? What is the$y$-intercept?

6. To graph the absolute value of a function, reflect any part for which $y < 0$ in the x-axis.

7. When solving absolute value equations you must consider both pieces of the absolute value function AND both restrictions, roots may be extraneous.

Notes to Self:

Challenge:

1. How do the graphs of $\frac{1}{x+1}$ and $\frac{x-1}{(x-1)(x+1)}$ differ?

2. What would the graph of $y = |x| - 4$ look like?